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Kolyvagin's "Euler Systems" in Cyclotomic Function Fields

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1. INTRODUCTION

In [K], Kolyvagin introduced "Euler systems" to prove important new results on ideal class groups of number fields. This method was developed by Rubin (see e.g. [R] and [R1]). In this paper, we consider the analogue of this method in cyclotomic function fields for cyclotomic units. Our approach follows Rubin's arguments in [R] and [R1] for number field cases quite closely. Notation is standard if not explained. Specifically let $k = F_q(T)$, where F_q is a field with q elements and T is a transcendental element over F_q, $R_T = F_q[T]$, $k(\Lambda_M)$ Mth cyclotomic function field where M is a monic polynomial in R_T and Λ_M is Mth torsion Carliz-Drinfeld module, λ_M a fixed generator of Λ_M, F the maximal real subfield of $k(\Lambda_M)$, A the integral closure of R_T in F, U the group of units of A, J be the set of monic irreducible polynomials which split completely in F, S be the set of square-free polynomials which are products of elements in J, N a positive integer with $(N, q(q-1)) = 1$, $S_N = \{a \in S : a$ is divisible only by monic prime polynomials Q with $N \mid \Phi(Q)\}$ where Φ is the Euler function, $G_R = Gal(F(\Lambda_R)/F) \cong Gal(k(\Lambda_R)/k)$ for every $R \in S$, $N_R = \sum_{\tau \in G_R} \tau \in Z[G_R]$ the norm operator, σ_Q a fixed generator of G_Q and $D_Q = \sum_{i=1}^{\Phi(Q)-1} i\sigma_Q^i$ for a prime monic polynomial Q in S, $D_R = \prod_{Q \mid R} D_Q$ for any $R \in S$, $G = Gal(F/k)$, I the group of fractional ideals of F, I_Q the subgroup of I generated by the prime ideals above Q for a prime monic polynomial in R_T, $[y]$ and $[y]_Q$ the projections of principal ideal (y) in I/NI and I_Q/NI_Q respectively. We also use $\bar{a}$ to denote the class which a belongs to in the quotient group.

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2. "EULER SYSTEMS"

Let D be the subgroup of F^* generated by $\{\lambda^{q-1} \mid \lambda \in \Lambda_M - \{0\}\}$. For any $R \in S$, $Q \mid R$ with Q a monic prime polynomial and any $\eta \in D$, write $\eta = (\prod_\lambda \lambda^{q-1})/(\prod_{\lambda'} (\lambda')^{q-1})$. Put

$$\xi_R(\eta) = \left(\prod_\lambda N_{k(\Lambda_{MR})/F(\Lambda_R)}\left(\lambda - \sum_{P \mid R} \lambda_P\right)\right) \Big/ \left(\prod_{\lambda'} N_{k(\Lambda_{MR}/F(\Lambda_R)}\left(\lambda' - \sum_{P \mid R} \lambda_P\right)\right),$$

where P is a monic prime polynomial. Then

ES 1. $\xi_R(\eta) \in F(\Lambda_R)^*$.

ES 2. *$\xi_R(\eta)$ is a global unit if $R \neq 1$.*

ES 3. *$N_Q \xi_R(\eta) = (Fr_Q - 1)\, \xi_{R/Q}(\eta)$, where Fr_Q denotes the Frobenius of Q in $G_{R/Q}$.*

ES 4. *$\xi_R(\eta) \equiv \xi_{R/Q}(\eta)$ modulo every prime above Q.*

Proof. ES 1 and ES 2 can follow from [GR, Corollary 1.9], and ES 4 is obvious. We only need to check ES 3. Since

$$\begin{aligned} N_Q &\left(N_{k(\Lambda_{MR})/F(\Lambda_R)}\left(\lambda - \sum_{P \mid R} \lambda\right)\right) \\ &= N_{k(\Lambda_{M(R/Q)})/F(\Lambda_{R/Q})}\left(\left(\lambda^Q - \sum_{P \mid (R/Q)} \lambda_P^Q\right) \Big/ \left(\lambda - \sum_{P \mid (R/Q)} \lambda_P\right)\right). \end{aligned}$$

Note Q splits completely in F, so $(Q, M) = 1$ and

$$N_{k(\Lambda_{M(R/Q)})/F(\Lambda_{R/Q})}\left(\lambda^Q - \sum_{P \mid (R/Q)} \lambda_P^Q\right) = Fr_Q\left(N_{k(\Lambda_{M(R/Q)})/F(\Lambda_{R/Q})}\left(\lambda - \sum_{P \mid (R/Q)} \lambda_P\right)\right)$$

by [H, Corollary 2.5]. Then ES 3 follows.

Remark 1. Call $\{\xi_R(\eta)\}_{R \in S}$ an Euler system starting from η. This Euler system depends on not only η but also the representation forms of η.

By using an Euler system, we will obtain a sequence of elements in F^*.

LEMMA 2.1. *If $R \in S_N$ then $\overline{D_R(\xi_R(\eta))} \in [F(\Lambda_R)^*/(F(\Lambda_R)^*)^N]^{G_R}$.*

Proof. If $Q \mid R$, then

$$\begin{aligned}(\sigma_Q - 1)\, D_R(\xi_R(\eta)) &= (\Phi(Q) - N_Q)\, D_{R/Q}(\xi_R(\eta)) \\ &\equiv (1 - Fr_Q)\, D_{R/Q}(\xi_{R/Q}(\eta)) \qquad (\mathrm{mod}(F(\Lambda_R))^{\Phi(Q)}).\end{aligned}$$

Since $Fr_Q \in G_{R/Q}$, this lemma follows by induction.

LEMMA 2.2. *For every $R \in S_N$ there is a $\kappa_R \in F^*$ such that*

$$\kappa_R \equiv D_R(\xi_R(\eta)) \qquad (\mathrm{mod}(F(\Lambda_R)^*)^N).$$

Proof. Since the constant field does not change in geometric cyclotomic extensions of k (see [H, Prop. 5.2]) and $(N, q-1) = 1$, the Nth roots of unity are trivial in cyclotomic function fields. Therefore we can define a 1-cocycle C_R: $Gal(F(\Lambda_R)/F) \to F(\Lambda_R)^*$ by

$$C_R(\sigma) = [(\sigma - 1)\, D_R(\xi_R(\eta))]^{1/N}.$$

By Hilbert 90, there is a $\beta \in F(\Lambda_R)^*$ such that $c_R(\sigma) = (\sigma - 1)\,\beta$ for all $\sigma \in Gal(F(\Lambda_R)/F)$. Put $\kappa_R = D_R(\xi_R(\eta))/\beta^N$ desired.

Now we look at the factorization of κ_R in the group of fractional ideals of F.

LEMMA 2.3. *For a prime monic $Q \in S_N$, there is a unique G-equivariant surjection*

$$\varphi_Q\colon (A/QA)^* \to I_Q/NI_Q$$

which makes the following diagram commute:

$$\begin{tikzcd} & F(\Lambda_Q)^* \arrow[dl, "(1-\sigma_Q)\,x \leftarrow\!\shortmid\, x"'] \arrow[dr, "x \mapsto [N_Q x]_Q"] & \\ (A/QA)^* \arrow[rr, dashed, "\varphi_Q"'] & & I_Q/NI_Q. \end{tikzcd}$$

Proof. Because all primes above Q are totally, tamely ramified in $F(\Lambda_Q)/F$ and $(1 - \sigma_Q)\,\lambda_Q$ is a generator in $(R_T/(Q))^*$, the left vertical map is surjective by Chinese Remainder Theorem and the kernel of this map is given by the elements which have order divisible by $\Phi(Q)$ at all primes above Q. It is clear that the right vertical map is surjective and the kernel of left vertical map is contained in the kernel of right vertical map. This proves the lemma.

Remark 2. Put $a = x^{1-\sigma_Q}$, then $a^{\Phi(Q)/N} = 1$ if and only if x has order divisible by N at all primes above Q if and only if $[N_Q(x)]_Q = 0$ in I_Q/NI_Q. Therefore Ker $\varphi_Q = \{a \in (A/QA)^* \mid a^{\Phi(Q)/N} = 1\}$.

Remark 3. For Q as in Lemma 2.3 we will also write φ_Q for the induced homomorphism

$$\varphi_Q\colon \{y \in F^*/(F^*)^N\colon [y]_Q = 0\} \to I_Q/NI_Q.$$

LEMMA 2.4. *Suppose $R \in S_N$ with $R \neq 1$ and Q is a monic prime polynomial in R_T.*

(i) *If $(Q, R) = 1$ then $[\kappa_R]_Q = 0$.*

(ii) *If $Q \mid R$ then $[\kappa_R]_Q = \varphi_Q(\kappa_{R/Q})$.*

Proof. By Lemma 2.2, $\kappa_R \equiv D_R(\xi_R(\eta)) \pmod{(F(\Lambda_R)^*)^N}$.

(i) If $(Q, R) = 1$, all primes above Q are unramified in $F(\Lambda_R)/F$. So (i) follows from ES 2.

(ii) If $Q \mid R$, write $R = QT$, then $\kappa_R = D_R(\xi_R(\eta))/\beta_R^N$ with $\beta_R \in F(\Lambda_R)^*$ and $\kappa_T = D_T(\xi_T(\eta))/\beta_T^N$ with $\beta_T \in F(\Lambda_T)^*$. Furthermore

$$(\sigma - 1)\, \beta_R = [(\sigma - 1)\, D_R(\xi_R(\eta))]^{1/N}$$

for all $\sigma \in Gal(F(\Lambda_R)/F)$ and

$$(\sigma - 1)\, \beta_T = [(\sigma - 1)\, D_T(\xi_T(\eta))]^{1/N}$$

for all $\sigma \in Gal(F(\Lambda_T)/F)$. By (i), we also can assume that β_T is prime to Q. Since (β_R^N) is an ideal in F and all primes above Q are unramified in $F(\Lambda_R)/F(\Lambda_Q)$, there is a γ in $F(\Lambda_Q)$ such that $\beta_R \gamma^{\Phi(Q)/N}$ is a unit at all primes above Q. So $[N_Q \gamma]_Q = [\kappa_R]_Q$. Consider

$$\begin{aligned}
&(1 - \sigma_Q)(\gamma^{\Phi(Q)/N}) \\
&\quad \equiv (\sigma_Q - 1)\, \beta_R = [(\sigma_Q - 1)\, D_R(\xi_R(\eta))]^{1/N} \\
&\quad = [(\Phi(Q) - N_Q)\, D_T(\xi_R(\eta))]^{1/N} = (D_T \xi_R(\eta))^{\Phi(Q)/N}/(N_Q D_T(\xi_R(\eta)))^{1/N} \\
&\quad = (D_T \xi_R)^{\Phi(Q)/N}/[D_T(Fr_Q - 1)\, \xi_T(\eta)]^{1/N} \\
&\quad \equiv (D_T(\xi_T(\eta)))^{\Phi(Q)/N}/(Fr_Q - 1)\, \beta_T \\
&\quad \equiv (D_T(\xi_T(\eta)))^{\Phi(Q)/N}/\beta_T^{\Phi(Q)} = (D_T(\xi_T(\eta))/\beta_T^N)^{\Phi(Q)/N}.
\end{aligned}$$

modulo any prime above Q. By Lemma 2.3 and Remark 2, the lemma follows.

3. AN APPLICATION OF THE CHEBOTAREV THEOREM

Fix a rational prime l with $(l, q(q-1)) = 1$ and let C denote the l-part of the ideal class group of F. We have

THEOREM 3.1. *Suppose N is a power of l, $\mathscr{C} \in C$, W a finite G-submodule of $F^*/(F^*)^N$, and $\psi: W \to (Z/NZ)[G]$ a G-equivariant map. Then there are infinitely many primes ϱ of F such that*

(i) $\varrho \in \mathscr{C}$

(ii) $N \mid \Phi(Q)$ *and Q splits completely in F/k, where* $Q = \varrho \cap R_T$

(iii) $[w]_Q = 0$ *for all* $w \in W$, *and there is a* $u \in (Z/NZ)^*$ *such that* $\varphi_Q(w) = u\psi(w)\varrho$ *for all* $w \in W$.

Proof. Suppose H is the maximal unramified abelian l-extension of F in which all ∞ primes of F split completely in H/F, then C is identified with $Gal(H/F)$ by the Artin map (see [Ro]). Let $F' = F_{q^v}F$ be a constant field extension with $N \mid (q^v - 1)$, so $F'(W^{1/N})/F'$ is the Kummer extension. Since any prime of F lying above ∞ is unramified and inert in F'/F, $H \cap F' = F$. Write τ a generator of $Gal(F'/F) \cong Gal(F_{q^v}/F_q)$. By a simple computation, τ acting on $Gal(F'(W^{1/N})/F')$ is given by raising q-th power. On the other hand, HF' is abelian over F, τ acts trivially on $Gal(HF'/F')$. Every element in $Gal(F'(W^{1/N})F')$ has order dividing N but $(N, (q-1)) = 1$. Therefore $HF' \cap F'(W^{1/N}) = F'$.

If an element $\bar{a}$ in $F^*/(F^*)^N$ satisfies $a = b^N$ for some b in F', then there is a ε in $F^*_{q^v}$ such that $\tau(b) = \varepsilon b$. Let δ be a generator of $F^*_{q^v}$, then $\varepsilon^{-1} = \delta^m$ for some integer m. Note $\varepsilon^N = 1$ and $(N, (q-1)) = 1$, we have $(q^v - 1) \mid mN$ and $(q-1) \mid m$. By Chinese Remainder Theorem, there is an integer χ such that $Nx \equiv 0 \bmod (q^v - 1)$ and $(q-1)\, x \equiv m \bmod (q^v - 1)$. Since $\delta^{\chi} b \in F^*$ and

$$a = b^N = (\delta^{\chi} b)^N \in (F^*)^N.$$

Therefore $F^*/(F^*)^N \subseteq F'^*/(F'^*)^N$.

By the Kummer pairing, $Gal(F'(W^{1/N})/F') \cong Hom(W, F^*_{q^v})$. Fix a primitive Nth root of unity ζ_N and define a map $\imath: (Z/NZ)[G] \to F^*_{q^v}$ by $\imath(1_G) = \zeta_N$ and $\imath(g) = 1$ for $g \in G$, $g \neq 1_G$. Then $\imath\psi \in Hom(W, F^*_{q^v})$ and there is a $\gamma \in Gal(F'(W^{1/N})/F')$ such that $\imath\psi(w) = \gamma(w^{1/N})/w^{1/N}$ for all $w \in W$. Choose $\rho \in Gal(HF'(W^{1/N})/F)$ such that ρ restricts to γ on $F'(W^{1/N})$ and to $\mathscr{C}$ on H. By the Chebotarev theorem (see [W. P289, Theorem 12; P104, Corollary 2]), there are infinitely many primes ϱ of F which are unramified in $HF'(W^{1/N})/k$, have degree 1, and whose Frobenius class is the conjugacy class of ρ. Then we must verify that ϱ satisfies (i), (ii) and (iii).

(i) follows from the Artin map.

Write $Q = \varrho \cap R_T$. Since ρ is trivial on F', Q splits completely in $F_{q^v}k$ and $N \mid q^v - 1 \mid \Phi(Q)$. Note ϱ has modular degree one and Q is unramified in F/k, Q splits completely in F. This proves (ii).

Since ϱ is unramified in $F'(W^{1/N})/F$, $[w]_Q = 0$ for all $w \in W$. By Lemma 2.3

and Remark 2, $ord_{\varrho}(\varphi(w))=0$ if and only if w is an Nth power modulo ϱ. On the other hand, since Q splits completely in F/k,

$$ord_{\varrho}(\psi(w)\,\varrho)=0 \Leftrightarrow \iota\psi(w)=1 \Leftrightarrow \gamma(w^{1/N})=w^{1/N}.$$

By a simple computation, $\gamma(w^{1/N})=w^{1/N}$ if and only if a $\sigma\gamma\sigma^{-1}(w^{1/N})=w^{1/N}$ for all a $\sigma\in Gal(F'(W^{1/N})/F)$ if and only if w is an Nth power module ϱ. Therefore there is $u\in(Z/NZ)^*$ such that

$$ord_{\varrho}(\varphi_Q(w))=u\; ord_{\varrho}(\psi(w)\,\varrho)$$

for all $w\in W$. Note that G acts on all the primes above Q transitively, this proves (iii).

4. The Ideal Class Group of F

In this section we apply the above results to study the ideal class group of F. We adopt the same notation as before. Let l be a rational prime with $(l, q\Phi(M))=1$ and χ be irreducible Z_l-representation of G. Write $e(\chi)=(q-1)/\Phi(M)\sum_{\gamma\in G} Tr(\chi(\gamma))\,\gamma^{-1}\in Z_l[G]$ the χ-idempotent for every irreducible Z_l-representation of G, $C(\chi)=e(\chi)\,C$, $P=$ the subgroup of $k(\Lambda_M)^*$ generated F_q^* and the set $\Lambda_M-\{0\}$, $E=P\cap U$ the cyclotomic units of F, and $(U/E)(\chi)$ the χ-component of the l-sylow subgroup of the finite group U/E.

The following lemma guarantees the existence of Minkowsk's unit in function fields.

Lemma 4.1. *Suppose L/k is a finite Galois extension, and B is the integral closure of R_T in L. Then there is a unit ε of B such that the subgroup generated by $\{\varepsilon^{\sigma}\mid\sigma\in Gal(L/k)\}$ has finite index in the group of units of B.*

Proof. Let $\infty_1, ..., \infty_g$ be all infinity primes of L, D_{∞_1} be the decomposition group of ∞_1 and $\sigma_1, ..., \sigma_g$ be the representatives of cosets $Gal(L/k)/D_{\infty_1}$ with $\sigma_i(\infty_1)=\infty_i$, $1\leqslant i\leqslant g$. Since $Pic^0(L)$ is finite, we can find a unit ε of B such that ε is zero at ∞_1 and pole at ∞_i for all $i=2, ..., g$. Then $(v_{\infty_i}(\varepsilon^{\sigma_j}))_{1\leqslant i\leqslant g-1,\, 1\leqslant j\leqslant g-1}$ is invertible where v_{∞_i}, stands for the exponential valuation of ∞_i for $i=1, ..., g$ (see [Wa, Lemma 5.28]). Therefore ε is as desired.

Corollary. *$e(\chi)(U\otimes_z Z_l)$ is a free rank-1 $e(\chi)\,Z_l[G]$-module.*

Proof. Since $(l, q-1)=1$, $U\otimes_z Z_l$ is free Z_l-module with rank $\#(G)-1$ and $U\otimes_z Z_l=\bigoplus_\chi e(\chi)(U\otimes_z Z_l)$ where χ run out of all irreducible Z_l representations of G. By the above lemma, there is a Minkowski's unit ε in U such that $e(\chi)\,\varepsilon\neq 1$ if χ is nontrivial. Note $e(\chi)\,Z_l[G]$ is isomorphic to the ring of integers of the unramified extension of $\mathbb{Q}_l$ of degree $\dim(\chi)$ and $e(\chi)(U\otimes_z Z_l)$ can naturally be regarded as $e(\chi)\,Z_l[G]$-module, we obtain the result as desired by considering the Z_l-rank of U.

THEOREM 4.3. $\#(C(\chi)) = \#(U/E)(\chi)$ *for every irreducible* Z_l *representation* χ *of* G.

Proof. We may assume that χ is nontrivial. Put $N = l\#(U/E)(\chi)\#(C(\chi))$. Since

$$e(\chi)(U/U^N) \cong e(\chi)((U/U^N) \otimes_z Z_l) \cong e(\chi)((U \otimes_z Z_l)/(U^N \otimes_z Z_l))$$
$$\cong (e(\chi)(U \otimes_z Z_l))/(e(\chi)(U \otimes_z Z_l))^N$$

so $e(\chi)(U/U^N)$ is a cyclic $e(\chi)\, Z_l[G]$-module by the above corollary. Note $e(\chi)\, Z_l[G]$ is isomorphic to the ring of integers of the unramified extension of Q_l of degree $\dim(\chi)$, so there is a divisor t of N such that

$$(U/E)(\chi) \cong (e(\chi)(U/U^N))/(e(\chi)(E/(U^N)) \cong e(\chi)(Z/tZ)[G]$$

and $\#(U/E)(\chi) = t^{\dim(\chi)}$. Fix $\eta \in e(\chi)(U/U^N)$ such that η^{q-1} generates $e(\chi)(U/U^N)$ as $e(\chi)\, Z_l[G]$-module. Then η has order N and $\eta^t \in e(\chi)(E/U^N)$ Suppose $\mathscr{C}_1, ..., \mathscr{C}_i$ are the elements in $C(\chi)$ and we have chosen $\varrho_1, ..., \varrho_i$ such that the class of ϱ_j in $C(\chi)$ is $\mathscr{C}_j$ and $Q_j = \varrho_j \cap R_T$ splits completely in F and $N \mid \Phi(Q_j)$ for $j = 1, ..., i$. Write $R_i = \prod_{j \leqslant i} Q_j$ (so $R_0 = 1$), we have an Euler system starting from $\eta^{t(q-1)}$ by Remark 1 and obtain κ_{R_i} by Lemma 2.2. Let m_i be the order of $e(\chi)\, \kappa_{R_i}$ in $F^*/(F^*)^N$, $t_i = N/m_i$ and W be the G-module generated by $e(\chi)\, \kappa_{R_i}$. Define a G-equivariant map $\psi\colon W \to (Z/NZ)[G]$ by $\psi(e(\chi)\, \kappa_{R_i}) = t_i e(\chi)$. $\langle \mathscr{C}_1, ..., \mathscr{C}_i \rangle$ denotes the $e(\chi)\, Z_l[G]$-submodule of $C(\chi)$.

If $\langle \mathscr{C}_1, ..., \mathscr{C}_i \rangle = C(\chi)$, we are done.

Otherwise there is a $\mathscr{C}_{i+1} \in C(\chi) - \langle \mathscr{C}_1, ..., \mathscr{C}_i \rangle$. By Theorem 3.1, we have ϱ_{i+1} such that the class of ϱ_{i+1} in $C(\chi)$ is $\mathscr{C}_{i+1}$, $Q_{i+1} = \varrho_{i+1} \cap R_T$ splits completely in F, $N \mid \Phi(Q_{i+1})$ and there is $u \in (Z/NZ)^*$ satisfying $\varphi_{Q_{i+1}}(e(\chi)(\kappa_{R_i})) = u t_i e(\chi)\, \varrho_{i+1}$. Put $R_{i+1} = R_i Q_{i+1}$, we again have an Euler system starting from $\eta^{t(q-1)}$ by Remark 1 and obtain $\kappa_{R_{i+1}}$ by Lemma 2.2. Let m_{i+1} be the order of $e(\chi)\, \kappa_{R_{i+1}}$ in $F^*/(F^*)^N$ and $t_{i+1} = N/m_{i+1}$, then $e(\chi)(\kappa_{R_{i+1}}) = f^{t_{i+1}}$ for some $f \in F^*$. Consider

$$[e(\chi)(\kappa_{R_{i+1}})] = \varphi_{Q_{i+1}}(e(\chi)(\kappa_{Ri}) + \sum_{j \leqslant i} [e(\chi)(\kappa_{R_{i+1}})]_{Q_j} \equiv u t_i e(\chi)\, \varrho_{i+1}$$

in $I/(NI, I_{Q_1}, ..., I_{Q_i})$. So $t_{i+1} \mid t_i$. Note $\kappa_{R_0} = \kappa_1 = \eta^{t(q-1)}$, then $t_{i+1} \mid t = t_0$ by induction and $(N/t_{i+1})\, C(\chi) = 0$. Now we have

$$[f] \equiv u(t_i/t_{i+1})\, e(\chi)\, \varrho_{i+1}$$

in $I/((N/t_{i+1})\, I, I_{Q_1}, ..., I_{Q_i})$. Therefore

$$(t_i/t_{i+1})\, \mathscr{C}_{i+1} \equiv 0$$

in $C(\chi)/\langle \mathscr{C}_1, ..., \mathscr{C}_i \rangle$.

Continue until we obtain a set $\{\mathscr{C}_1, ..., \mathscr{C}_k\}$ which generates $C(\chi)$ as $e(\chi)\, Z_l[G]$-module. Then

$$[\langle \mathscr{C}_1, ..., \mathscr{C}_{i+1} \rangle : \langle \mathscr{C}_1, ..., \mathscr{C}_i \rangle] \mid (t_i/t_{i+1})^{\dim(\chi)}$$

for $i = 0, 1, ..., k-1$. Therefore

$$\# C(\chi) \,\Big|\, \prod_{i=1}^{k} (t_{i-1}/t_i)^{\dim(\chi)} \,\Big|\, t_0^{\dim(\chi)} = \# (U/E)(\chi).$$

By [GR, Theorem; Prop. 1.14], this proves the above theorem.

Remark 4. Theorem 4.3 can be regarded as a Gras's conjecture in function fields case. When M is a prime polynomial and $l =$ characteristic of F, the same result was proved by Goss (see [G, Prop. 2.15]).

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